MINISTRY OF EDUCATION, SINGAPORE
in collaboration with
CAMBRIDGE ASSESSMENT INTERNATIONAL EDUCATION
General Certificate of Education Advanced Level
Higher 3

MATHEMATICS

9820/01

Paper 1

For examination from 2025

SPECIMEN PAPER

3 hours

Additional Materials: Printed Answer Booklet
List of Formulae and Results (MF27)

READ THESE INSTRUCTIONS FIRST

Answer **all** questions.
Write your answers on the Printed Answer Booklet. Follow the instructions on the front cover of the answer booklet.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are **not** allowed in a question, you must present the mathematical steps using mathematical notations and not calculator commands.
You must show all necessary working clearly.

The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **6** printed pages and **2** blank pages.

Singapore Examinations and Assessment Board

Cambridge Assessment International Education

1 **(a)** Show that $y = x$ is a solution of the differential equation

$$y^2 + yx - x^2 - x^2\frac{dy}{dx} = 0.$$ [1]

(b) Prove that the differential equation

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

can be transformed into the differential equation

$$x\frac{du}{dx} = F(u) - u$$

by using the substitution $u = \frac{y}{x}$. [2]

(c) A solution curve of the differential equation

$$y^2 + yx - x^2 - x^2\frac{dy}{dx} = 0$$

passes through the point (1, 2). Find the equation of the curve. [8]

2 The integral I_n, where n is a non-negative integer, is defined by $I_n = \int_0^{\frac{\pi}{3}} \tan^n \theta \, d\theta$.

(a) Show that, for $n \geqslant 2$,

$$I_n = \frac{3^{\frac{n-1}{2}}}{n-1} - I_{n-2}.$$ [5]

(b) Find the exact values of I_5 and I_6. [5]

3 **(a)** **(i)** For all positive real numbers x, y and z, prove that

$$\tfrac{1}{2}\left[\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2\right] \geqslant \frac{x}{z}.$$ [2]

(ii) Hence, for all positive real numbers x, y and z, prove that

$$\left(\frac{x}{y}\right)^2 + \left(\frac{y}{z}\right)^2 + \left(\frac{z}{x}\right)^2 \geqslant \frac{x}{z} + \frac{y}{x} + \frac{z}{y} \geqslant 3.$$ [4]

(b) **(i)** Let $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ be two non-zero vectors. By considering the scalar product of **a** and **b**, or otherwise, prove that

$$\left(\sum_{i=1}^{3} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{3} a_i^2\right)\left(\sum_{i=1}^{3} b_i^2\right).$$ [3]

(ii) Hence, for all positive real numbers x, y and z, prove that

$$x + y + z \leqslant 2\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right).$$ [5]

A graphing calculator must not be used in question 4.

4 The functions f and g are defined on the real numbers by

$$f(x) = x^3 - 3x + 1,$$

$$g(x) = \frac{1}{1-x}, \text{ for } x \neq 1.$$

(a) Show that $f(x) = 0$ has three distinct real roots. [2]

Let α, β and γ be the roots of $f(x) = 0$, where $\alpha < \beta < \gamma$.

(b) Prove that $g(\alpha) = \beta$, $g(\beta) = \gamma$ and $g(\gamma) = \alpha$. [5]

(c) Given that h is a quadratic function such that

$$h(\alpha) = \beta,\ h(\beta) = \gamma \text{ and } h(\gamma) = \alpha,$$

find $h(x)$. [5]

5 An ordering of the numbers 1 to n such that no number i is in position i is called a *derangement*. For example, 2 3 4 1 is a derangement for $n = 4$ whereas 2 3 1 4 is not because 4 is in position 4.

(a) Write down all of the derangements for $n = 4$. [2]

(b) Use the principle of inclusion and exclusion to prove that the number of derangements of the numbers 1 to n, D_n, is

$$n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right).$$ [5]

(c) Show that $\left|D_n - \frac{n!}{\mathrm{e}}\right| < \frac{1}{1+n}$. Deduce that D_n is the closest integer to $\frac{n!}{\mathrm{e}}$. [4]

(d) Show that the probability that a randomly generated ordering of the numbers 1 to n is a derangement tends to $\frac{1}{\mathrm{e}}$ as $n \to \infty$. [2]

Please turn over.

Use the information in the mathematical text to answer Question 6. You should read the whole mathematical text before you start answering the questions.

Combinatorial Interpretation of the Harmonic Numbers

The harmonic numbers are the partial sums of the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$

The nth harmonic number is $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$.

The first five harmonic numbers are $H_1 = 1$, $H_2 = \frac{3}{2}$, $H_3 = \frac{11}{6}$, $H_4 = \frac{25}{12}$, $H_5 = \frac{137}{60}$.

The harmonic series diverges, since H_n increases without bound, but it does so very slowly. For example, $H_{1\,000\,000} < 15$.

Although H_n is never an integer for $n > 1$, it can be expressed as a rational number whose numerator and denominator have a combinatorial significance.

Specifically, for $n \geqslant 1$ we can always write

$$H_n = \frac{p_n}{n!}$$

where $p_1 = 1$, and for $n > 1$,

$$p_n = np_{n-1} + (n-1)!. \qquad (1)$$

There is a familiar combinatorial interpretation for $n!$ and we seek a combinatorial interpretation for p_n.

For integers $n \geqslant k \geqslant 1$, let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the number of ways for n distinct people to sit around k identical circular tables where no tables are allowed to be empty.

We can compute the numbers $\begin{bmatrix} n \\ k \end{bmatrix}$ recursively using

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)! \text{ and for } k > 1, \begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n\begin{bmatrix} n \\ k \end{bmatrix}. \qquad (2)$$

By comparing (1) and (2) it follows that

$$H_n = \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}.$$

So H_n is equal to the number of ways of arranging $n + 1$ people around 2 identical circular tables (with neither table empty) divided by the number of ways of arranging n people in a row.

6 **(a)** By summing the areas of appropriate rectangles defined using the graph of $y = \frac{1}{x}$, for $x > 0$, prove that

$$\frac{1}{n} + \ln n < H_n < 1 + \ln n.$$ [3]

(b) Deduce that the harmonic series diverges and that $H_{1\,000\,000} < 15$. [2]

(c) Prove that for $n > 1$, $p_n = np_{n-1} + (n-1)!$. [3]

(d) **(i)** Prove that $\begin{bmatrix} n \\ 1 \end{bmatrix} = (n-1)!$. [2]

(ii) Prove that for $k > 1$, $\begin{bmatrix} n+1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + n\begin{bmatrix} n \\ k \end{bmatrix}$. [3]

(e) Deduce that for $n > 1$

$$H_n = \frac{1}{n!}\begin{bmatrix} n+1 \\ 2 \end{bmatrix}.$$ [3]

For $n > 1$, let the positive integer k be such that $2^k \leqslant n < 2^{k+1}$, and let $\mathrm{M}(n)$ be the product of all the positive odd integers $\leqslant n$.

(f) Show that for $n > 1$, $2^k \times \mathrm{M}(n) \times H_n$ is an odd integer. [2]

(g) Deduce that for $n > 1$, H_n is not an integer. [2]

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